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LETTER TO THE EDITOR

On parasupersymmetric Hamiltonians and vector mesons in magnetic fields

N Debergh†

Theoretical and Mathematical Physics, Institute of Physics, B5, University of Liège, B-4000-Liège 1, Belgium

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Abstract. A new Sakata–Taketani equation describing vector mesons in interaction with a constant magnetic field is proposed. It leads to real energies as opposed to relativistic equations, which have already been pointed out, associated with this particular interacting context. In particular, the connection between this new proposal and a specific model coming from parasupersymmetric quantum mechanics is studied.

Relativistic descriptions of vector mesons have been considered for a long time. For example, it is well known that *free* vector mesons are specified through the Kemmer equation [1]

$$\{\beta^\mu p_\mu - m\}\psi(x) = 0 \tag{1}$$

where, in the system of natural units, we refer to Minkowskian events $x = \{x^\mu\} = \{t, \boldsymbol{x}\}$ through the metric tensor $G = \{g^{\mu\nu} \mid g^{oo} = -g^{ii} = 1\}$. Moreover, the (10×10) matrices β_μ generate the Kemmer algebra

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = g_{\mu\nu}\beta_\lambda + g_{\lambda\nu}\beta_\mu. \tag{2}$$

The extension to the *interacting* context is not so easy. Indeed, the study of vector mesons in external electromagnetic fields has been associated [2] with

$$\left\{\beta^\mu\Pi_\mu - m + (1-\beta_5^2)\frac{e}{2m}S_{\mu\nu}F^{\mu\nu}\right\}\psi(x) = 0 \tag{3}$$

generalizing (1) through the minimal coupling characterized by $\Pi_\mu = p_\mu - eA_\mu$, in terms of the fourpotential $A(x) = \{A_\mu(x)\}$ and completed by the anomalous coupling in terms of the electromagnetic field $F = (\boldsymbol{E}, \boldsymbol{B}) = \{F_{\mu\nu}\}$, where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{4}$$

Here the matrix β_5 is defined through [3]

$$\beta_5 = \frac{\mathrm{i}}{4}\varepsilon_{\mu\nu\rho\sigma}\beta^\mu\beta^\nu\beta^\rho\beta^\sigma \qquad \varepsilon_{0123} = 1 \tag{5}$$

† Chercheur Institut Interuniversitaire des Sciences Nucléaires, Bruxelles.

while the spin tensor is given by

$$S_{\mu\nu} = \mathrm{i}[\beta_\mu, \beta_\nu]. \tag{6}$$

The relativistic equation (3), restricted to the constant magnetic field case (along the x_3 direction), can also be presented on the so-called Sakata–Taketani form [4]

$$\begin{aligned} \mathrm{i}\frac{\partial}{\partial t}\varphi(x) &= H^{\mathrm{ST}}\varphi(x) \\ &= \left\{ m(I\otimes\sigma_2) + \frac{1}{2m}\Pi^2 I\otimes(\sigma_2+\mathrm{i}\sigma_1) - \frac{\mathrm{i}}{m}S_j S_l \Pi_j \Pi_l \otimes \sigma_1 + \frac{eB}{m}S_3\otimes\sigma_2 \right\}\varphi(x) \end{aligned} \tag{7}$$

taking account only of the (six) physical components, in terms of direct products of $D^{(1)}$ and Pauli-matrices. Going to the eigenvalues of the square of this Sakata–Taketani Hamiltonian, we obtain

$$E^2 = m^2 + 2eB(n+\tfrac{1}{2}) + 2eBs \qquad n = 0, 1, 2, \ldots, s = 0, \pm 1 \tag{8}$$

as is easily verified if we limit ourselves to the so-called perpendicular part (i.e. in the plane (x_1, x_2)). The key point is to notice that the non-relativistic limit corresponding to (8)

$$H_{\mathrm{NR}} = \frac{1}{2m}(\Pi_1^2 + \Pi_2^2) + \frac{eB}{m}S_3 \tag{9}$$

with eigenvalues

$$E_{\mathrm{NR}} = \omega(n+\tfrac{1}{2}) + \omega\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \omega = \frac{eB}{m} \tag{10}$$

is actually the parasupersymmetric Hamiltonian pointed out by Rubakov and Spiridonov [5]. A specific feature of this parasupersymmetric model is the existence of negative eigenvalues (associated here with $n = 0$ and $s = -1$). This evidently leads to complex relativistic energies for sufficiently large magnetic fields, as is clear from (8), confirming the results already noticed by Tsai [6].

We propose here to eliminate this defect in the relativistic point of view by using another parasupersymmetric model. Indeed, it is known that the Beckers–Debergh (BD) parasupersymmetric oscillator [7] is independent of the Rubakov–Spiridonov oscillator and, in particular, is characterized exclusively by positive energy eigenvalues. More precisely, the BD spectrum corresponds to (compare with (10))

$$E_{\mathrm{NR}} = \omega(n+\tfrac{1}{2}) + \frac{\omega}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{11}$$

The remaining point is just to provide in evidence a Sakata–Taketani Hamiltonian whose non-relativistic limit would lead to such a spectrum. We thus suggest considering

$$\begin{aligned} \mathcal{H}^{\mathrm{ST}} &= m(I\otimes\sigma_2) + \frac{1}{2m}(\Pi_1^2+\Pi_2^2)I\otimes(\sigma_2 - \mathrm{i}\sigma_1) + \frac{\mathrm{i}}{2m}(\Pi_1^2+\Pi_2^2)S_3^2\otimes\sigma_1 \\ &\quad - (\Pi_1^2+\Pi_2^2)\lambda + (\pi_1^2+\pi_2^2)\lambda - \frac{\mathrm{i}}{2m}(\Pi_1\pi_1 - \Pi_2\pi_2)(S_1^2 - S_2^2)\otimes\sigma_1 \\ &\quad - \frac{\mathrm{i}}{2m}(\Pi_1\pi_2 + \Pi_2\pi_1)\{S_1, S_2\}\otimes\sigma_1 + \frac{eB}{m}\eta. \end{aligned} \tag{12}$$

The quantities $\Pi_a = p_a - eA_a$ $(a = 1, 2)$ have already been characterized through the minimal substitution, while the operators $\pi_a = p_a + eA_a$ come directly from Johnson–Lippmann [8] developments. It is sufficient, in order to justify (12), to notice that, in the *free* context, we have $\Pi_a = \pi_a = p_a$, ensuring that $\mathcal{H}^{\mathrm{ST}}$ then reduces to the well known free Sakata–Taketani Hamiltonian.

It also has to be understood that only the operators $(1/2m)(\Pi_1^2+\Pi_2^2)$, $(1/2m)(\pi_1^2+\pi_2^2)$ (corresponding to two independent oscillators), as well as their products, have known eigenvalues. Then, squaring the Hamiltonian (12), we are led to the following constraints

$$\{\lambda, (S_1^2 - S_2^2) \otimes \sigma_1\} = \{\lambda, \{S_1, S_2\} \otimes \sigma_1\} = 0 \tag{13a}$$

$$-\frac{\mathrm{i}}{2m}\{(S_1^2 - S_2^2) \otimes \sigma_1, \eta\} + 2(\{S_1, S_2\} \otimes \sigma_1)\lambda - \frac{\mathrm{i}}{2m}\{S_1, S_2\} \otimes \sigma_3 = 0 \tag{13b}$$

$$-\frac{\mathrm{i}}{2m}\{\{S_1, S_2\} \otimes \sigma_1, \eta\} - 2((S_1^2 - S_2^2) \otimes \sigma_1)\lambda + \frac{\mathrm{i}}{2m}(S_1^2 - S_2^2) \otimes \sigma_3 = 0. \tag{13c}$$

Next, in order to take the non-relativistic limit, we have to impose

$$(\mathcal{H}^{\mathrm{ST}})^2 = m^2 + 2mH_{\mathrm{NR}} \otimes I. \tag{14}$$

Finally, eliminating the contribution of $(\Pi_1^2 + \Pi_2^2)^2$, $(\pi_1^2 + \pi_2^2)^2$ and $(\Pi_1^2 + \Pi_2^2)(\pi_1^2 + \pi_2^2)$, we obtain

$$E_{\mathrm{NR}} = \begin{pmatrix} E_{\mathrm{NR}}^{(1)} & 0 & 0 \\ 0 & E_{\mathrm{NR}}^{(2)} & 0 \\ 0 & 0 & E_{\mathrm{NR}}^{(3)} \end{pmatrix} \tag{15}$$

where

$$\begin{aligned} E_{\mathrm{NR}}^{(1)} = E_{\mathrm{NR}}^{(3)} &= \omega(n_1 + n_2 + 1)\left(\frac{1}{2} + \omega\left(a_3 + \frac{\mathrm{i}}{2m}\alpha_4\right)\right) + \frac{\omega^2}{8m} + \frac{\omega}{2}(4a_3m + 2\mathrm{i}\alpha_4) \\ &\quad + \frac{\omega^2}{2m}(\alpha_1^2 - \alpha_4^2 - \alpha_3^2 + 4\mathrm{i}a_3\alpha_4 m + 4a_1^2m^2) \end{aligned} \tag{16a}$$

and

$$\begin{aligned} E_{\mathrm{NR}}^{(2)} &= \omega(n_1 + \tfrac{1}{2})(1 - \mathrm{i}a_2m)\left(1 - \mathrm{i}\frac{\omega}{m}\alpha_6\right) + \omega(n_2 + \tfrac{1}{2})\mathrm{i}a_2m\left(1 - \mathrm{i}\frac{\omega}{m}\alpha_6\right) \\ &\quad + \frac{\omega}{2}(-\mathrm{i}\alpha_6 + \mathrm{i}\alpha_2) + \frac{\omega^2}{2m}(\alpha_5^2 + \alpha_2\alpha_6). \end{aligned} \tag{16b}$$

Here the eigenvalues $\omega(n_1 + \frac{1}{2})$ and $\omega(n_2 + \frac{1}{2})$ are those of the two independent oscillators $(1/2m)(\Pi_1^2 + \Pi_2^2)$ and $(1/2m)(\pi_1^2 + \pi_2^2)$, respectively. Moreover, the parameters appearing in (16) come from the forms

$$\lambda = \begin{pmatrix} a_1 & 0 & 0 & -\mathrm{i}/4m & a_3 & 0 \\ 0 & a_1 & 0 & -a_3 & -\mathrm{i}/4m & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2 \\ \mathrm{i}/4m & a_3 & 0 & -a_1 & 0 & 0 \\ -a_3 & \mathrm{i}/4m & 0 & 0 & -a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{17a}$$

$$\eta = \begin{pmatrix} \alpha_1 & -2\mathrm{i}a_1m & 0 & \alpha_4 & \alpha_3 & 0 \\ 2\mathrm{i}a_1m & \alpha_1 & 0 & -\alpha_3 & \alpha_4 & 0 \\ 0 & 0 & \alpha_5 & 0 & 0 & \alpha_2 \\ 4\mathrm{i}ma_3 - \alpha_4 & \alpha_3 & 0 & -\alpha_1 & 2\mathrm{i}a_1m & 0 \\ -\alpha_3 & 4\mathrm{i}ma_3 - \alpha_4 & 0 & -2\mathrm{i}a_1m & -\alpha_1 & 0 \\ 0 & 0 & \alpha_6 & 0 & 0 & -\alpha_5 \end{pmatrix} \tag{17b}$$

together with the constraints

$$a_1^2 - a_3^2 = -\frac{1}{16m^2} \qquad a_3\alpha_3 = a_1\alpha_1. \tag{18}$$

A specific choice for satisfying these conditions in particular is

$$\begin{aligned} &a_1 = 0 \qquad a_2 = -\frac{\mathrm{i}}{2m} \qquad a_3 = \frac{1}{4m} \\ &\alpha_1 = \mathrm{i}\frac{m}{\omega}\sqrt{3} \qquad \alpha_2 = \mathrm{i} \qquad \alpha_3 = 0 \\ &\alpha_4 = -\mathrm{i}\frac{m}{\omega} + \frac{\mathrm{i}}{2} \qquad \alpha_5 = \mathrm{i}\frac{m}{\omega} \qquad \alpha_6 = \mathrm{i}\frac{m}{\omega} \end{aligned} \tag{19}$$

leading to

$$\begin{aligned} &E_{\mathrm{NR}}^{(1)} = E_{\mathrm{NR}}^{(3)} = \omega(n_1 + n_2 + 1) \equiv \omega(n' + 1) \\ &E_{\mathrm{NR}}^{(2)} = \omega(n_1 + n_2) \equiv \omega n' \end{aligned} \tag{20}$$

i.e. to the BD parasupersymmetric spectrum. This evidently corresponds to

$$\lambda = -\frac{\mathrm{i}}{4m} I \otimes (\sigma_1 + \mathrm{i}\sigma_2) + \frac{\mathrm{i}}{4m} S_3^2 \otimes \sigma_1 + \frac{\mathrm{i}}{4m} S_3 \otimes \sigma_1 \tag{21a}$$

and

$$\begin{aligned} \eta = {} & \mathrm{i}\frac{m}{\omega}\sqrt{3} S_3^2 \otimes \sigma_3 + \mathrm{i}\frac{m}{\omega}(1 - S_3^2) \otimes \sigma_3 + \frac{m}{\omega} S_3^2 \otimes \sigma_2 + \frac{\mathrm{i}}{2} S_3^2 \otimes \sigma_1 \\ & + \frac{\mathrm{i}}{2}(1 - S_3^2) \otimes (\sigma_1 + \mathrm{i}\sigma_2) + \mathrm{i}\frac{m}{2\omega}(1 - S_3^2) \otimes (\sigma_1 - \mathrm{i}\sigma_2). \end{aligned} \tag{21b}$$

Then, a Sakata–Taketani Hamiltonian characterized by real energies and associated with the BD parasupersymmetric oscillator is given by

$$\begin{aligned} \mathcal{H}^{\mathrm{ST}} = {} & m\left[I \otimes \sigma_2 + \mathrm{i}I \otimes \sigma_3 + \mathrm{i}(\sqrt{3} - 1) S_3^2 \otimes \sigma_3 + \frac{\mathrm{i}}{2} I \otimes (\sigma_1 - \mathrm{i}\sigma_2) - \frac{\mathrm{i}}{2} S_3^2 \otimes (\sigma_1 + \mathrm{i}\sigma_2) \right] \\ & + \frac{1}{4m}(\Pi_1^2 + \Pi_2^2)[-\mathrm{i}S_3 \otimes \sigma_1 + \mathrm{i}S_3^2 \otimes \sigma_1 + I \otimes (\sigma_2 - \mathrm{i}\sigma_1)] \\ & + \frac{1}{4m}(\pi_1^2 + \pi_2^2)[\mathrm{i}S_3 \otimes \sigma_1 + \mathrm{i}S_3^2 \otimes \sigma_1 + I \otimes (\sigma_2 - \mathrm{i}\sigma_1)] \\ & - \frac{\mathrm{i}}{2m}(\Pi_1\pi_1 - \Pi_2\pi_2)(S_1^2 - S_2^2) \otimes \sigma_1 - \frac{1}{2m}(\Pi_1\pi_2 + \Pi_2\pi_1)\{S_1, S_2\} \otimes \sigma_1 \\ & + \frac{eB}{m}\left[\frac{\mathrm{i}}{2} I \otimes (\sigma_1 + \mathrm{i}\sigma_2) + \frac{1}{2} S_3^2 \otimes \sigma_2 \right]. \end{aligned} \tag{22}$$

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